4.2 - Reduction of Order

Consider a 2nd-order DE

$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ (1)

If y_1, y_2 form a fundamental set of solutions, then they are lin. indep.

So if $c_1 y_1 + c_2 y_2 = 0$ with $c_1, c_2 \neq 0$,

we can write $y_2 = \boxed{-\frac{c_1}{c_2}} y_1$,

where $-\frac{c_1}{c_2}$ is a function of x.

Then $y_2 = u(x) y_1$ for some unknown function of x $u(x)$.

In standard form, (1) becomes

$y'' + P(x)y' + Q(x)y = 0$ by dividing by $a_2(x)$

If y_1 is a known solution, let's consider

$y_2 = u(x)y_1$. $y_2' = u'y_1 + uy_1'$ and

$y_2'' = u''y_1 + u'y_1' + u'y_1' + u y_1''$

$y_2'' = u''y_1 + 2u'y_1' + u y_1''$

substitute:

$u''y_1 + 2u'y_1' + uy_1'' + P(u'y_1 + uy_1') + Q(uy_1) = 0$

$\underbrace{u(y_1'' + Py_1' + Qy_1)}_{=0} + u''y_1 + (2y_1' + Py_1)u' = 0$

(y_1 is a known solution to the DE)

$u''y_1 + (2y_1' + Py_1)u' = 0$ (2nd order)

Let $w = u'$. Then $w' = u''$ and we have

$\frac{dw}{dx}$ $w'y_1 + (2y_1' + Py_1)w = 0$ (1st order)

(this is linear, separable)

$dw\, y_1 + (2y_1' + Py_1)w\, dx = 0$

$$\frac{dw}{w} = -\left(\frac{2y_1'}{y_1} + P\right)dx = \left(-\frac{2y_1'}{y_1} - P\right)dx$$

$$\ln|w| = -2\ln|y_1| - \int P(x)\,dx$$

$$e^{\ln|wy_1^2|} = e^{-\int P(x)\,dx}$$

$$wy_1^2 = e^{-\int P(x)\,dx}$$

$$u' = w = \frac{e^{-\int P(x)\,dx}}{y_1^2}$$

The reduction of order formula is

$y_2 = y_1 u$

if the DE is in std form

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}dx$$

Contrast: For a linear DE, $\mu = e^{\int P(x)dx}$

Example: The indicated function $y_1(x)$ is a solution of the given differential equation. Use reduction of order (the process or the formula) to find a second solution $y_2(x)$.

$y'' + 9y = 0; \quad y_1 = \sin 3x$

process: $y_2 = u y_1 \Rightarrow y_2 = u \sin 3x$

$$y_2' = u' \sin 3x + 3u \cos 3x$$

$$y_2'' = u'' \sin 3x + 6u' \cos 3x - 9u \sin 3x$$

$$u'' \sin 3x + 6u' \cos 3x - \cancel{9u \sin 3x} + \cancel{9u \sin 3x} = 0$$

$$u'' \sin 3x + 6u' \cos 3x = 0$$

Let $w = u'$: $w' \sin 3x + 6w \cos 3x = 0$

Linear, separable: $w' + 6 \dfrac{\cos 3x}{\sin 3x} w = 0$

$$\mu = e^{\int 6 \frac{\cos 3x}{\sin 3x} dx} = e^{2\int \frac{3\cos 3x}{\sin 3x} dx}$$

$$\mu = e^{2 \ln|\sin 3x|} = \sin^2 3x$$

$$\frac{d}{dx}(\sin^2 3x \, w) = 0$$

$$\sin^2 3x \, w = c \Rightarrow w = c \csc^2 3x$$

$$u = c \int \csc^2 3x \, dx = -\frac{1}{3} c \cot 3x + D$$

$y_2 = u y_1 \Rightarrow y_2 = \left(-\frac{1}{3} c \cot 3x + D\right) y_1$ gets subsumed by

$y = c_1 y_1 + c_2 y_2$ gets subsumed by $y_2 = \cot 3x \sin 3x$

$$y_2 = \cos 3x$$

Example: $6y'' + y' - y = 0; \quad y_1 = e^{x/3}$

Formula: $y_2 = y_1 \displaystyle\int \frac{e^{-\int P dx}}{y_1^2} dx$

DE must be in std form: $y'' + \frac{1}{6}y' - \frac{1}{6}y = 0$

$P = \frac{1}{6} \Rightarrow \int P\,dx = \frac{1}{6}x$

$$e^{-\int P dx} = e^{-\frac{1}{6}x} \Rightarrow \frac{e^{-\int P dx}}{y_1^2} = \frac{e^{-\frac{1}{6}x}}{e^{2x/3}}$$

$$\int \frac{e^{-\int P dx}}{y_1^2}\,dx = \int e^{-5x/6}\,dx = \cancel{-\frac{6}{5}}e^{-5x/6} \cancel{+C}$$

$$y_2 = y_1\left(e^{-5x/6}\right) = e^{x/3}e^{-5x/6} = e^{-x/2}$$

$$y_2 = e^{-x/2}$$

General solution: $y = c_1 e^{x/3} + c_2 e^{-x/2}$

Example: $x^2y'' - 3xy' + 5y = 0; \quad y_1 = x^2\cos(\ln x)$

Example: The indicated function $y_1(x)$ is a solution of the associated homogeneous equation. Use the method of reduction of order [the process] to find a second solution $y_2(x)$ of the homogeneous equation and a particular solution $y_p(x)$ of the given nonhomogeneous equation.

$y'' - 4y' + 3y = x; \quad y_1 = e^x$

use the process for nonhomogeneous DEs.

$y_2 = ue^x \quad y_2' = u'e^x + ue^x, \quad y_2'' = u''e^x + 2u'e^x + ue^x$

$u''e^x + 2u'e^x + \cancel{ue^x} - 4u'e^x - \cancel{4ue^x} + \cancel{3ue^x} = x$

$w = u'$: $\quad w'e^x - 2we^x = x \qquad$ (linear)

$$w' - 2w = xe^{-x} \qquad \mu = e^{-2x}$$

$$\frac{d}{dx}(e^{-2x} w) = \underbrace{xe^{-3x}}_{\text{integration by parts}}$$

$$e^{-2x} w = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C_1$$

$$u' = w = -\frac{1}{3}xe^{-x} - \frac{1}{9}e^{-x} + C_1 e^{2x}$$

$$u = \frac{1}{3}xe^{-x} + \frac{4}{9}e^{-x} + Ce^{2x}$$

$$y_2 = u\,y_1 = \underbrace{\boxed{\frac{1}{3}x + \frac{4}{9}}}_{y_p} + C\underbrace{\boxed{e^{3x}}}_{y_2}$$

$$y_2 = e^{3x} \qquad y_p = \frac{1}{3}x + \frac{4}{9}$$

General solution: $y = \underbrace{c_1 e^{x} + c_2 e^{3x}}_{y_c} + \underbrace{\frac{1}{3}x + \frac{4}{9}}_{y_p}$